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The cone spanned by Cohen-Macaulay modules and applications

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1 Introduction

This is a joint work [3] with H. Dao (Univ. of Kansas).

On a smooth projective variety, we can define the intersection number for a given divisor and a given curve. By this pairing, we can define the numerical equivalence on divisors. We get a (finitely generated) lattice if we divide the set of Weil divisors by the numerical equivalence. In order to study the intersection pairing, we have some concepts of "positive" divisors, e.g., ample, base point-free, nef, etc.. Consider the cone spanned by positive elements in the lattice tensored with the field of real numbers. The cone gives us many informations on the given algebraic variety.

In this note, we are interested in the intersection pairing around a fixed singular point of a scheme, or the vertex of the affine cone of a smooth projective variety. Let R be a Noetherian (Cohen-Macaulay) local ring corresponding to the given point. We first define a pairing between a finitely generated module, and a module of finite length and finite projective dimension. Consider the Grothendieck group of finitely generated R-modules, and divide it by the numerical equivalence. Then, we get a finitely generated lattice. It is natural to think that Cohen-Macaulay modules are positive elements under the pairing. So, we study the cone spanned by Cohen-Macaulay modules in the numerical Grothendieck group tensored with $\mathbb{R}$.

2 Intersection pairing on $\operatorname{Spec} R$ and the Cohen-Macaulay cone

We always assume that R is a d-dimensional Noetherian Cohen-Macaulay local domain such that one of the following conditions are satisfied[1]:

(a) R is a homomorphic image of an excellent regular local ring containing $\mathbb{Q}$.

(b) R is essentially of finite type over a field, $\mathbb{Z}$ or a complete DVR.

In this note, modules are always assumed to be finitely generated.

Let $G_0(R)$ be the Grothendieck group of finitely generated R-modules. The symbol $[M]$ means the element in $G_0(R)$ corresponding to an R-module M. Let $C(R)$ be the category of modules of finite length and finite projective dimension. Here, note that $R/(x_1, \ldots, x_d) \in C(R)$ for a system of parameters $x_1, \ldots, x_d$. In particular, $C(R)$ is not empty.[2] For $L \in C(R)$, we define

$$\chi_L : G_0(R) \longrightarrow \mathbb{Z} \quad \text{by} \quad \chi_L([M]) = \textstyle\sum_i (-1)^i \ell_R(\mathrm{Tor}_i^R(L, M)).$$

Consider the map

$$C(R) \times G_0(R) \to \mathbb{Z} \quad \text{defined by} \quad (L, [M]) \mapsto \chi_L([M]). \tag{1}$$

Here, we define *numerical equivalence* as follows. For $\alpha, \beta \in G_0(R)$,

$$\alpha \equiv \beta \overset{\text{def}}{\Longleftrightarrow} \chi_L(\alpha) = \chi_L(\beta) \ \text{ for any } \ L \in C(R).$$

Here, we put

$$\overline{G_0(R)} = G_0(R)/\{\alpha \in G_0(R) \mid \alpha \equiv 0\}.$$

By Theorem 3.1 and Remark 3.5 in [7], we have the following result.

Theorem 1 *$\overline{G_0(R)}$ is a finitely generated torsion-free abelian group.*

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[1] If either (a) or (b) is satisfied, there exists a regular alteration of $\operatorname{Spec} R$ by de Jong's theorem [5].

[2] By the new intersection theorem due to Roberts, we know that, for a Noetherian local ring R, $C(R)$ is not empty if and only if R is Cohen-Macaulay.

Remark 2 Maximal Cohen-Macaulay modules behave as positive elements under the pairing (1) as follows.

Let L be an object in $C(R)$. Then, by Auslander-Buchsbaum formula, we have

$$\operatorname{depth} L + \operatorname{pd}_R L = \operatorname{depth} R = d.$$

Then, we have $\operatorname{pd}_R L = d$. Let $\mathbb{F}.$ be the minimal free resolution of L. Then, it is very easy to check that the complex $\mathbb{F}.$ has a depth sensitive property, i.e., for any module N, we have

$$\operatorname{depth} N = d - \max\{i \mid H_i(\mathbb{F}. \otimes_R N) \neq 0\}.$$

We say that M is a MCM (maximal Cohen-Macaulay) module if $\operatorname{depth} M = d$. By the depth sensitivity, if M is MCM, then $\operatorname{Tor}_i^R(L, M) = 0$ for any $i > 0$. Therefore, we have

$$\chi_L([M]) = \ell_R(L \otimes_R M) > 0.$$

By Auslander-Buchsbaum formula, any MCM module over a regular local ring is free. We say that a ring R is of finite (Cohen-Macaulay) representation type if there are only finitely many isomorphism classes of indecomposable MCM's. If R is of finite representation type, then R has only isolated singularity. It was proved that a Gorenstein local ring of finite representation type has a simple singularity. Simple singularities are of finite representation type. We refere the reader to Yoshino [9] for the representation theory of MCM's.

Bad Cohen-Macaulay rings have many MCM's in general. But, if we do not assume that R is Cohen-Macaulay, it is not known whether there exists an MCM module. This open problem is called the small Mac conjecture [4].

Example 3

1. If $L = R/(x_1, \ldots, x_d)$ for a system of parameters $x_1, \ldots, x_d$, then $\chi_L([R]) \neq 0$. Hence, $\overline{G_0(R)} \neq 0$.

2. If $d \leq 2$, then $\operatorname{rank} \overline{G_0(R)} = 1$. See Proposition 3.7 in [7].

3. Let X be a smooth projective variety with embedding $X \hookrightarrow \mathbb{P}^n$. Let R (resp. D) be the affine cone (resp. the very ample divisor) of this

embedding. Then, we have the following commutative diagram:

$$\begin{array}{ccccc} G_0(R)_{\mathbb{Q}} & \xrightarrow{\sim} & A_*(R)_{\mathbb{Q}} & \xleftarrow{\sim} & CH^{\cdot}(X)_{\mathbb{Q}}/D\cdot CH^{\cdot}(X)_{\mathbb{Q}} \\ \downarrow & & \downarrow & & \downarrow \\ \overline{G_0(R)}_{\mathbb{Q}} & \xrightarrow{\sim} & \overline{A_*(R)}_{\mathbb{Q}} & \xleftarrow{\phi} & CH^{\cdot}_{num}(X)_{\mathbb{Q}}/D\cdot CH^{\cdot}_{num}(X)_{\mathbb{Q}} \end{array}$$

(a) By the commutativity of this diagram, ϕ is a surjection. Therefore, we have

$$\operatorname{rank}\overline{G_0(R)} \leq \dim_{\mathbb{Q}} CH^{\cdot}_{num}(X)_{\mathbb{Q}}/D\cdot CH^{\cdot}_{num}(X)_{\mathbb{Q}}. \tag{2}$$

(b) If $CH^{\cdot}(X)_{\mathbb{Q}} \simeq CH^{\cdot}_{num}(X)_{\mathbb{Q}}$, then ϕ is an isomorphism ([7], [8]). In this case, the equality holds in (2).

(c) There exists an example such that ϕ is not an isomorphism [8].

Further, Roberts and Srinivas [8] proved the following: Assume that the standard conjecture and Bloch-Beilinson conjecture are true. Then ϕ is an isomorphism if the defining ideal of R is generated by polynomials with coefficients in the algebraic closure of the prime field.

4. It is conjectured that $\overline{G_0(R)}_{\mathbb{Q}} \simeq \mathbb{Q}$ if R is complete intersection isolated singularity with d even.

It is true if R is the affine cone of a smooth projective variety X over $\mathbb{C}$ ([2]). In fact, since we have an injection

$$CH^i_{hom}(X)_{\mathbb{Q}} \longrightarrow H^{2i}(X, \mathbb{Q}) = \mathbb{Q}$$

and the natural surjection

$$CH^i_{hom}(X)_{\mathbb{Q}} \longrightarrow CH^i_{num}(X)_{\mathbb{Q}} \neq 0,$$

we know $CH^i_{num}(X)_{\mathbb{Q}} = \mathbb{Q}$ for each $i = 0, 1, \ldots, \dim X$. Here, remark that $H^{2i}(X, \mathbb{Q}) = \mathbb{Q}$ since the dimension of X is odd. Then, we have

$$CH^{\cdot}_{num}(X)_{\mathbb{Q}}/D\cdot CH^{\cdot}_{num}(X)_{\mathbb{Q}} = \mathbb{Q}.$$

Therefore, the rank of $\overline{G_0(R)}$ is one by 3 (a) as above.

Definition 4 *We define the Cohen-Macaulay cone as follows:*

$$C_{CM}(R) = \sum_{M:MCM} \mathbb{R}_{\geq 0}[M] \subset \overline{G_0(R)}_{\mathbb{R}}.$$

Here $\overline{G_0(R)}_{\mathbb{R}} = \overline{G_0(R)} \otimes_{\mathbb{Z}} \mathbb{R}$*.*

We refer the reader to [1] for basic properties on Cohen-Macaulay cones. It is easy to see that the dimension of the cone is equal to the rank of $\overline{G_0(R)}$. Further, we have

$$\overline{G_0(R)}_{\mathbb{R}} \supset C_{CM}(R)^- \supset C_{CM}(R) \supset Int(C_{CM}(R)^-) = Int(C_{CM}(R)) \ni [R],$$

where $C_{CM}(R)^-$ is the closure of $C_{CM}(R)$ with respect to the classical topology on $\overline{G_0(R)}_{\mathbb{R}}$, and $Int(-)$ is the interior.

If R is of finite representation type, then $C_{CM}(R)$ is a strongly convex polyhedral cone, in particular $C_{CM}(R)^- = C_{CM}(R)$.

We have no example that $C_{CM}(R)^-$ is not equal to $C_{CM}(R)$, or $C_{CM}(R)$ is not a polyhedral cone.

Remark that, for any $L \in C(R)$, χ_L induces $\overline{\chi_L}$ which makes the following diagram commutative:

$$\begin{array}{ccc} G_0(R) & \xrightarrow{\chi_L} & \mathbb{Z} \\ \downarrow & \nearrow \overline{\chi_L} & \\ \overline{G_0(R)} & & \end{array}$$

The map $\overline{\chi_L}$ induces

$$(\overline{\chi_L})_{\mathbb{R}} : \overline{G_0(R)}_{\mathbb{R}} \longrightarrow \mathbb{R}.$$

Let $x_1, \ldots, x_d$ be a system of parameters. Consider the map

$$\overline{\chi_{R/(\underline{x})}} : \overline{G_0(R)} \longrightarrow \mathbb{Z}.$$

Let $\mathbb{K}.$ be the Koszul complex with respect to $\underline{x}$. This map satisfies

$$\overline{\chi_{R/(\underline{x})}}([M]) = \operatorname{rank} M \cdot \overline{\chi_{R/(\underline{x})}}([R]),$$

since $\mathbb{K}.$ is the minimal free resolution of $R/(\underline{x})$ and $\mathbb{K}.$ admits this property. Therefore, we have a map

$$\operatorname{rk} : \overline{G_0(R)} \longrightarrow \mathbb{Z}$$

and

$$\mathrm{rk}_{\mathbb{R}} : \overline{G_0(R)_{\mathbb{R}}} \longrightarrow \mathbb{R}$$

defined by $\mathrm{rk}([M]) = \mathrm{rank}\, M$. (Here, $\mathrm{rk} = \frac{1}{\chi_{R/(\underline{x})}([R])} \overline{\chi_{R/(\underline{x})}}$.)

Let F be the kernel of the map rk. Then, F is generated by cycles $[M]$ with $\dim M < d$. Thus, we have

$$\overline{G_0(R)} = \mathbb{Z}[R] \oplus F \quad \text{and} \quad \overline{G_0(R)_{\mathbb{R}}} = \mathbb{R}[R] \oplus F_{\mathbb{R}}.$$

Example 5 1. Put $R = k[x,y,z,w]_{(x,y,z,w)}/(xy - zw)$, where k is a field. Then, $F = \mathbb{Z}[R/(x,z)] \simeq \mathbb{Z}$. This ring has only three indecomposable maximal Cohen-Macaulay modules, R, (x,z) and (x,w).

Then, the Cohen-Macaulay cone is spanned by

$$[(x,z)] = ([R], -[R/(x,z)]) \quad \text{and} \quad [(x,w)] = ([R], [R/(x,z)])$$

in $\overline{G_0(R)_{\mathbb{R}}} = \mathbb{R}[R] \oplus F_{\mathbb{R}}$.

2. Put $R = k[x_1, x_2, \dots, x_6]_{(x_1,x_2,\dots,x_6)}/(x_1x_2 + x_3x_4 + x_5x_6)$, where k is a field. Then, $F = \mathbb{Z}[R/(x_1,x_3,x_5)] \simeq \mathbb{Z}$. This ring has only three indecomposable maximal Cohen-Macaulay modules, R, M_1 and M_2, where M_1 and M_2 are maximal Cohen-Macaulay modules of rank 2.

   Then, the Cohen-Macaulay cone is spanned by

$$[M_1] = (2[R], [R/(x_1,x_3,x_5)]) \quad \text{and} \quad [M_2] = (2[R], -[R/(x_1,x_3,x_5)])$$

   in $\overline{G_0(R)_{\mathbb{R}}} = \mathbb{R}[R] \oplus F_{\mathbb{R}}$.

   The Cohen-Macaulay cone of this ring is not spanned by classes of modules of rank one.

3. Put $R = k[x,y,z,w]_{(x,y,z,w)}/(xy - f_1 f_2 \cdots f_t)$, where k is an algebraically closed field of characteristic zero. Here, we assume that f_1, f_2, $\dots$, f_t are pairwise coprime linear forms in $k[z,w]$. In this case, we have

$$F = (\oplus_i \mathbb{Z}[R/(x,f_i)]) / \mathbb{Z}([R/(x,f_1)] + \cdots + [R/(x,f_t)]) \simeq \mathbb{Z}^{t-1}.$$

   We can prove that the Cohen-Macaulay cone is minimally spanned by the following $2^t - 2$ MCM's of rank one.

$$\{\{(x, f_{i_1} f_{i_2} \cdots f_{i_s}) \mid 1 \le s < t, \quad 1 \le i_1 < i_2 < \cdots < i_s \le t\}$$

   This ring is of finite representation type if and only if $t \le 3$.

We shall give the classical topology to $\overline{G_0(R)}_{\mathbb{R}}$. The following is the main result. By the result, we know that $C_{CM}(R)^-$ is a strongly convex cone, that is, $C_{CM}(R)^-$ does not contain a line through the origin.

Theorem 6 *Let r be a positive integer. Then,* $(\mathrm{rk}_{\mathbb{R}})^{-1} \cap C_{CM}(R)^-$ *is a compact set.*

Corollary 7 *Assume that R is a Cohen-Macaulay local domain. Then, for any positive integer r,*

$$\{[M] \in \overline{G_0(R)} \mid M \text{ is a maximal Cohen-Macaulay module of rank } r\ \}$$

is a finite subset of $\overline{G_0(R)}$.

Further, assume that R is a normal domain. Then, we have the *determinant map* (or the *first Chern class map*) $c_1 : G_0(R) \to A_{d-1}(R)$.

We can also define numerical equivalence on $A_{d-1}(R)$. Then, we define the class group modulo numerical equivalence to be

$$\overline{A_{d-1}(R)} = A_{d-1}(R)/\equiv .$$

By Proposition 3.7 and Example 4.1 in [7], we know that it is also a finitely generated torsion-free abelian group.

Here we can prove that there exists the map $\overline{c_1}$ which makes the following diagram commutative:

$$\begin{array}{ccc} G_0(R) & \xrightarrow{c_1} & A_{d-1}(R) \\ \downarrow & & \downarrow \\ \overline{G_0(R)} & \xrightarrow{\overline{c_1}} & \overline{A_{d-1}(R)} \end{array}$$

By the commutativity of the above diagram, we have the following:

Corollary 8 *Let R be a d-dimensional Cohen-Macaulay local normal domain. Assume that*

(*) *the kernel of the natural map $A_{d-1}(R) \longrightarrow \overline{A_{d-1}(R)}$ is a finite group.*

Then, for any positive integer r,

$$\{c_1([M]) \in A_{d-1}(R) \mid M \textit{ is a maximal Cohen-Macaulay module of rank } r\ \}$$

is a finite subset of $A_{d-1}(R)$.

In particular, R has only finitely many maximal Cohen-Macaulay modules of rank one up to isomorphism.

Assume that R is a standard graded Cohen-Macaulay domain over a field of characteristic zero. If R has an isolated singularity with $\dim R \geq 3$, then it is proved that R has only finitely many maximal Cohen-Macaulay modules of rank one up to isomorphism. This result is essentially written in Karroum [6].

Theorem 9 (Dao-Kurano, [2]) *Let R be a 3-dimensional isolated hypersurface singularity with desingularization. Then, the natural map*

$$A_2(R) \longrightarrow \overline{A_2(R)}$$

is an isomorphism. In particular (*) in Corollary 8 is satisfied. Therefore R has only finitely many MCM's of rank one.

Here, remark that an isolated hypersurface singularity of dimension d is factorial if $d \geq 4$. In this case, R is the only one MCM of rank one. If R is the affine cone of an elliptic curve, it has infinitely many MCM's of rank one.

Remark 10 *Put $B = \oplus_{n\geq 0} B_n = \mathbb{C}[B_1] = \mathbb{C}[y_0, y_1, \ldots, y_n]/I$, $R = B_{B_+}$, and $X = \mathrm{Proj}(B)$. Assume that X is smooth over $\mathbb{C}$. (Since $\dim R = d$, $\dim X = d-1$.)*

$$\begin{array}{ccccc}
\mathrm{CH}^1(X) & \longrightarrow & \mathrm{CH}^1(X)/c_1(\mathcal{O}_X(1))\mathrm{CH}^0(X) & = & A_{d-1}(R) \\
\downarrow & & \downarrow f & & \\
\mathrm{CH}^1_{\mathrm{num}}(X) & \longrightarrow & \mathrm{CH}^1_{\mathrm{num}}(X)/c_1(\mathcal{O}_X(1))\mathrm{CH}^0_{\mathrm{num}}(X) & \xrightarrow{g} & \overline{A_{d-1}(R)}
\end{array}$$

1. *Assume that R is a Cohen-Macaulay local normal ring with $d \geq 3$. Then, $\mathrm{CH}^1(X)$ is finitely generated and $f \otimes \mathbb{Q}$ is an isomorphism.*

2. *Assume that the ideal I is generated by some elements in $\overline{\mathbb{Q}}[y_0, y_1, \ldots, y_n]$. If some famous conjectures (the standard conjecture and Bloch-Beilinson conjecture) are true, then $g \otimes \mathbb{Q}$ is an isomorphism. (Roberts-Srinivas [8])*

Therefore, if R is a Cohen-Macaulay local normal ring with $d \geq 3$ such that X is defined over $\overline{\mathbb{Q}}$, and if some conjectures are true, then (*) is satisfied.

It is also proved in the case of positive characteristic.

If we remove the assumption that X is defined over $\overline{\mathbb{Q}}$, then there exists an example that $g \otimes \mathbb{Q}$ is not an isomorphism (Roberts-Srinivas [8]).

3 Proof of Theorem 6

First, we prove the following claim. By this claim, we know that the closure of the Cohen-Macaulay cone is strongly convex.

Claim 11 *We have $C_{CM}(R)^{-} \cap F_{\mathbb{R}} = \{0\}$.*

Proof. Let $\{e_1, \dots, e_s\}$ be a free basis of F. Regarding $\{[R], e_1, \dots, e_s\}$ as a normal orthogonal basis, we think that $\overline{G_0(R)_{\mathbb{R}}}$ is a metric space. For a vector v in $\overline{G_0(R)_{\mathbb{R}}}$, we denote $||v||$ the length of v.

Assume the contrary. Take $0 \neq \alpha \in C_{CM}(R)^{-} \cap F_{\mathbb{R}}$. We may assume that $||\alpha|| = 1$.

Then there exists a sequence of maximal Cohen-Macaulay modules M_1, M_2, ..., M_n, ...such that

$$\lim_{n \to \infty} \frac{[M_n]}{||[M_n]||} = \alpha \tag{3}$$

in $\overline{G_0(R)_{\mathbb{R}}}$.

Let x_1, ..., x_d be a system of parameters of R. Since M_n is a maximal Cohen-Macaulay module,

$$\ell_R(M_n/mM_n) \leq \ell_R(M_n/(\underline{x})M_n) = \mathrm{e}_{(\underline{x})}(M_n) = \operatorname{rank} M \cdot \mathrm{e}_{(\underline{x})}(R) \tag{4}$$

where $\mathrm{e}_{(\underline{x})}(-)$ denotes the multiplicity with respect to the ideal $(\underline{x})$. Put $e = \mathrm{e}_{(\underline{x})}(R)$ and $r_n = \operatorname{rank} M$ for each n. If $e = 1$, then R is a regular local ring, and therefore, $F = 0$. The assertion is obvious in this case. Suppose $e \geq 2$. We have an exact sequence of the form

$$0 \longrightarrow N_n \longrightarrow R^{r_n e} \longrightarrow M_n \longrightarrow 0$$

by (4). Remark that N_n is a maximal Cohen-Macaulay module.

Put $[M_n] = (r_n[R], m_n) \in \mathbb{Z}[R] \oplus F = \overline{G_0(R)}$. Then, we have

$$[N_n] = r_n e[R] - [M_n] = (r_n(e-1)[R], -m_n) \in \mathbb{Z}[R] \oplus F = \overline{G_0(R)}.$$

By (3), we have $\lim_{n\to\infty} \frac{r_n[R]}{||[M_n]||} = 0$ and $\lim_{n\to\infty} \frac{m_n}{||[M_n]||} = \alpha$. Hence, we have

$$\lim_{n\to\infty} \frac{r_n(e-1)[R]}{||[M_n]||} = 0 \tag{5}$$

$$\lim_{n\to\infty} \frac{-m_n}{||[M_n]||} = -\alpha. \tag{6}$$

On the other hand, we have

$$\frac{||[N_n]||}{||[M_n]||} \leq \frac{||r_n(e-1)[R]|| + ||m_n||}{||[M_n]||} = \frac{||r_n(e-1)[R]||}{||[M_n]||} + \frac{||m_n||}{||[M_n]||} \longrightarrow ||\alpha|| = 1.$$

Since $e \geq 2$, $||[N_n]|| \geq ||[M_n]||$. Hence,

$$\lim_{n\to\infty} \frac{||[N_n]||}{||[M_n]||} = 1. \tag{7}$$

Then, by (5), (6) and (7), we have $\lim_{n\to\infty} \frac{r_n(e-1)[R]}{||[N_n]||} = 0$ and $\lim_{n\to\infty} \frac{-m_n}{||[N_n]||} = -\alpha$. Therefore, $\lim_{n\to\infty} \frac{[N_n]}{||[N_n]||} = -\alpha$. Then, $-\alpha \in C_{CM}(R)^-$.

Let $L \in C(R)$. Then, $\chi_L(C_{CM}(R)) \subset \mathbb{R}_+$. Hence, $\chi_L(C_{CM}(R)^-) \subset \mathbb{R}_{\geq 0}$. Since $\pm\alpha \in C_{CM}(R)^-$, $\chi_L(\alpha) = 0$. By definition, we have $\alpha = 0$. It is a contradiction. We have completed the proof of Claim 11.

Now, we start to prove Theorem 7. Assume the contrary. Suppose that there exist infinitely many maximal Cohen-Macaulay modules L_1, L_2, $\ldots$, L_n, $\ldots$ such that

- $\operatorname{rank} L_n = r$ for all $n > 0$,
- $[L_i] \neq [L_j]$ in $\overline{G_0(R)}$ if $i \neq j$.

Put $[L_n] = (r[R], \ell_n) \in \mathbb{Z}[R] \oplus F = \overline{G_0(R)}$ for $n = 1, 2, \ldots$. Then, we have $\lim_{n\to\infty} ||\ell_n|| = \infty$ since $\ell_i \neq \ell_j$ for $i \neq j$. Put $S = \{v \in F_{\mathbb{R}} \mid ||v|| = 1\}$. Then $\frac{\ell_n}{||\ell_n||} \in S$ if $\ell_n \neq 0$. Since S is compact, $\{\ell_n/||\ell_n||\}_n$ contains a subsequence that converges to a point of S, say β. Suppose that $\lim_{n\to\infty} \frac{\ell_n}{||\ell_n||} = \beta$. Then, it is easy to check that $\lim_{n\to\infty} \frac{L_n}{||L_n||} = \beta \in S \subset F_{\mathbb{R}}$. Therefore, we have $0 \neq \beta \in C_{CM}(R)^- \cap F_{\mathbb{R}}$. It contradicts to Claim 11.

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